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CONVERGING CYLINDRICAL AND SPHERICAL DETONATION WAVES

(SKHODIASHCHIESIA TSILINDRICHESKIE I SPHERICHESKIE DETONATSIONNYE VOLNY)

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1. Spherical and cylindrical detonation waves, converging respectively to a point or to the axis of symmetry, are investigated. The usual assumptions are made :

1) the detonation wave is strong, i. e. the values of the pressure and internal energy in the undisturbed fluid can be neglected in comparison with their values in the disturbed gas ;

2) during the passage of the shock wave through the medium there is instantaneously released an energy Q, in $m^2/\sec^2$ (the magnitude of Q refers to unit mass) ;

3) the process in the disturbed fluid is polytropic with exponent γ.

From the conditions for the conservation of mass, momentum, and energy at the detonation wave [1 and 2] we have

$$v_2 = \beta D, \qquad p_2 = \beta \rho_1 D^2, \qquad \rho_2 = \rho_1/(1-\beta) \tag{1.1}$$

$$\beta = \frac{1}{\gamma+1}\left\{1 + \left[1 - 2(\gamma-1)(\gamma+1)\frac{Q}{D^2}\right]^{1/2}\right\} = \frac{\alpha}{\gamma+1} \tag{1.2}$$

The equations for the one-dimensional motion of a gas have the form, in Eulerian variables,

$$\rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial r} + \frac{\partial p}{\partial r} = 0, \qquad \frac{\partial \rho}{\partial t} + \rho\frac{\partial v}{\partial r} + v\frac{\partial \rho}{\partial r} + \frac{\nu\rho v}{r} = 0 \tag{1.3}$$

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial r} + \gamma p\frac{\partial v}{\partial r} + \frac{\gamma\nu p v}{r} = 0$$

Here $\nu = 0$ for the plane case, $\nu = 1$ for the cylindrical one, and $\nu = 2$ for spherical symmetry.

2. We investigate the case when the spherical or cylindrical detonation front converges from infinity according to the rule

$$r_2 = a_n(-t)^n \tag{2.1}$$

The motion is defined by the parameters a_n, ρ, r, t, γ and ν. From dimensional analysis [3] it follows that in this case the motion will be self-similar with a single dimensionless independent variable

$$\lambda = r/r_2\ (t) \tag{2.2}$$

As dimensionless functions we take the ratios of the unknown functions v, ρ, p to their values at the detonation front

$$-\omega(\lambda) = \frac{v}{v_2}, \qquad \delta(\lambda) = \frac{\rho}{\rho_2}, \qquad \pi(\lambda) = \frac{p}{p_2} \tag{2.3}$$

$$D = \frac{dr_2}{dt} = -na_n(-t)^{n-1} \tag{2.4}$$

From the conditions (1.1), (1.2) at the detonation front,

$$-v_2 = a_n\beta n(-t)^{n-1}, \qquad \rho_2 = \rho_1/(1-\beta), \qquad p_2 = \rho_1 a_n^2 \beta n^2(-t)^{2(n-1)} \tag{2.5}$$

We will further assume that β = const in the convergence process. This condition will be examined below. Converting Equations (1.3) to the dimensionless variables in (2.2) and (2.3), we get a system of three ordinary differential equations

$$\begin{gathered}\delta\omega'(\lambda+\beta\omega) + (1-\beta)\pi' - \delta\omega(n-1)/n = 0\\ \delta'(\lambda+\beta\omega) + \beta\delta\omega' + \nu\beta\delta\omega/\lambda = 0\\ \pi'(\lambda+\beta\omega) + \beta\gamma\pi\omega' - 2\pi(n-1)/n + \nu\beta\gamma\pi\omega/\lambda = 0\end{gathered} \tag{2.6}$$

with boundary conditions

$$\omega(1) = -1, \qquad \delta(1) = 1, \qquad \pi(1) = 1 \tag{2.7}$$

How we can determine n, will be examined in Section 4.

3. When the detonation front converges from infinity, the motion is defined by the parameters Q, ρ_1, r, t, γ and ν.

For dimensional considerations the detonation front must travel with a constant velocity $r_2 = a_1(-t)$, i.e. in Equation (2.1) it is necessary to put $n = 1$, where $a_1 \sim \sqrt{Q}$.

It is obvious that β = const for this case. Equations (2.6) are written in the form (for $n = 1$)

$$\begin{gathered}\delta\omega'(\lambda+\beta\omega) + (1-\beta)\pi' = 0\\ \delta'(\lambda+\beta\omega) + \beta\delta\omega' + \nu\beta\delta\omega/\lambda = 0\\ \pi'(\lambda+\beta\omega) + \beta\gamma\pi\omega' + \nu\beta\gamma\pi\omega/\lambda = 0\end{gathered} \tag{3.1}$$

The boundary conditions will be

$$\omega(1) = -1, \qquad \delta(1) = 1, \qquad \pi(1) = 1 \tag{3.2}$$

It is obvious that

$$\pi = \delta^\gamma \tag{3.3}$$

will be the integral of system (3.1) (analogous to the isentropic integral).

Substituting (3.3) into (3.1), we get a system of equations relating δ and ω

$$\begin{gathered}\delta(\lambda+\beta\omega)\omega' + \gamma(1-\beta)\delta^{\gamma-1}\delta' = 0\\ \beta\delta\omega' + (\lambda+\beta\omega)\delta' + \nu\beta\delta\omega/\lambda = 0\end{gathered} \tag{3.4}$$

Solving the system (3.4) for δ' and ω', we have

$$\frac{d\omega}{d\lambda} = \frac{\nu\beta\gamma(1-\beta)\delta^\gamma\omega}{\lambda[\delta(\lambda+\beta\omega)^2 - \gamma\beta(1-\beta)\delta^\gamma]}, \qquad \frac{d\delta}{d\lambda} = -\frac{\nu\beta\gamma\delta^2\omega(\lambda+\beta\omega)}{\lambda[\delta(\lambda+\beta\omega)^2 - \gamma\beta(1-\beta)\delta^\gamma]} \tag{3.5}$$

Furthermore

$$\delta^{\gamma-2}\frac{d\delta}{d\omega}=-\frac{\lambda+\beta\omega}{\gamma(1-\beta)} \tag{3.6}$$

The following can be said about the field of the integral curves of Equations (3.5) (Fig. 1):

1) The lines $\lambda = 0$ and $\omega = 0$ are integral curves;

2) Along the line $\lambda + \beta\omega = 0$ we have

$$\frac{d\delta}{d\lambda}=0, \frac{d\delta}{d\omega}=0\,(\delta=\text{const}), \frac{d\omega}{d\lambda}=-\frac{\nu}{\beta} \tag{3.7}$$

3) Along the line $\omega = 0$ we have

$$\frac{d\delta}{d\lambda}=0, \qquad \delta=\delta_0=\text{const} \tag{3.8}$$

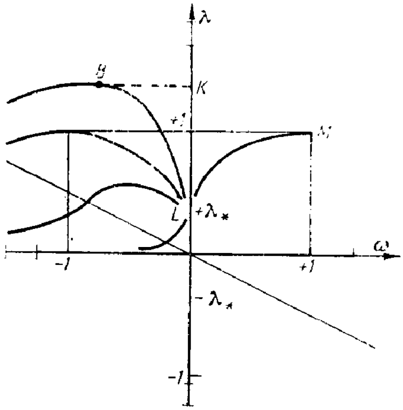

Fig. 1

It is easily seen, considering (3.6) and the boundary conditions (3.2), that

$$\delta_0^{\gamma-1}=\frac{2-\beta(\gamma+1)}{2\gamma(1-\beta)} \tag{3.9}$$

On the line $\omega = 0$ we have a singular point with coordinate

$$\lambda_*^2=\gamma\beta(1-\beta)\delta_0^{\gamma-1}.$$

Considering (1.2) and (3.9),

$$\lambda_*^2=\frac{\alpha(2-\alpha)}{2(\gamma+1)} \tag{3.10}$$

Hence it is evident that, even for arbitrary choice of α (giving up the Chapman-Jouguet condition), it is impossible for the singular point to have the coordinate $\lambda_* > 1$.

It is evident from the field of the integral curves (Fig. 1) that there is no integral curve going from the point $\lambda = 1$, $\omega = -1$, $\delta = 1$ to infinity along λ (a jump like BK gives an expansion shock, and introducing such an additional jump in order to obtain a solution is not permissible.

Thus, it is concluded that a self-similar solution does not exist for the case of detonation waves converging with a constant velocity (the isentropic case), even if the Chapman-Jouguet condition is given up.

This result was obtained by Landau and Staniukovich for the Chapman-Jouguet condition [4].

However, for divergent detonation waves with constant velocity, the solution exists (the curve LM), as was first discovered by Ia. B. Zel'dovich [1 and 2].

4. Dimensional analysis does not help in determining the law for the converging detonation front. We return to Equation (1.2), in which $\alpha = 2$ corresponds satisfactorily to a shock wave without the release of energy ($Q = 0$); $\alpha = 1$ corresponds to the Chapman-Jouguet detonation regime, where the wave travels along characteristics in the disturbed gas.

In those cases when $D^2 \gg Q$ (the internal energy of the matter in the detonation front is much greater than the released chemical energy, which corresponds to the sufficiently accelerating detonation wave), we can assume $\alpha = 2$ and use the self-similar solution of Guderley-Landau-Staniukovich [4].

We investigate the other extreme case (which is physically unreal), where $\alpha = 1$ in the convergence process, which insures the Chapman-Jouguet condition and the detonation wave travels along the characteristic. It is necessary for this that the energy yielded by the front Q be proportional to D^2 (or p_2 and T_2), i. e. Q must increase as the detonation front accelerates. (The internal energy of the matter in the detonation front is of the same order as the released chemical energy Q.)

While Q is not taken into account when using the Guderley-Landau-Staniukovich solution, its influence is now overestimated.

In this way we will seek the law for the convergence of the front as a characteristic dividing the disturbed and undisturbed media.

We use the equations of steady, one-dimensional motion in Lagrangian variables

$$\rho_1 \frac{\partial^2 u}{\partial t^2} = -\frac{\partial p}{\partial r}, \qquad \rho_1 = \frac{\rho(1+u_r)}{1+u/r}, \qquad \frac{p}{p_2} = \left(\frac{\rho}{\rho_2}\right)^\gamma \tag{4.1}$$

Here u is the displacement of a particle. The conditions on the detonation wave in the Chapman-Jouguet regime are

$$v_2 = \frac{D}{\gamma+1}, \qquad \rho_2 = \frac{\rho_1(\gamma+1)}{\gamma}, \qquad p_2 = \frac{\rho_1 D^2}{\gamma+1} \tag{4.2}$$

We put the first equation in (4.1) in the form

$$\frac{\partial^2 u}{\partial t^2} = a^2(r, u, u_r)\frac{\partial^2 u}{\partial r^2} + b(r, u, u_r) \tag{4.3}$$

Then the equations for the characteristics and the conditions on them are

$$dr = \pm a\, dt, \qquad du_t = \pm a\, du_r + b\, dt \tag{4.4}$$

From the third equation in (4.1), using the second equation and the relations (4.2), we obtain

$$p = \frac{1}{\gamma+1}\left(\frac{\gamma}{\gamma+1}\right)^\gamma \rho_1 \frac{D^2}{(1+u_r)^\gamma(1+u/r)^{\nu\gamma}} \tag{4.5}$$

Differentiating (4.5) and substituting into the first equation in (4.1) we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\gamma+1}\left(\frac{\gamma}{\gamma+1}\right)^\gamma \left\{\frac{\gamma D^2}{(1+u_r)^{\gamma+1}(1+u/r)^{\nu\gamma}}\frac{\partial^2 u}{\partial r^2} + \right.$$
$$\left. + \frac{\nu\gamma D^2}{(1+u_r)^\gamma(1+u/r)^{\nu\gamma+1}}\frac{u_r r - u}{r^2} - \frac{2D}{(1+u_r)^\gamma(1+u/r)^{\nu\gamma}}\frac{dD}{dr}\right. \tag{4.6}$$

Equating (4.6) and (4.3) we obtain

$$a^2(r, u, u_r) = \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{D^2}{(1+u_r)^{\gamma+1}(1+u/r)^{\nu\gamma}} \tag{4.7}$$

$$b(r, u, u_r) = \frac{1}{\gamma+1}\left(\frac{1}{\gamma+1}\right)^\gamma \left[\frac{\nu\gamma D^2(u_r r - u)}{(1+u_r)^\gamma(1+u/r)^{\nu\gamma+1}r^2} - \frac{2D}{(1+u_r)^\gamma(1+u/r)^{\nu\gamma}}\frac{dD}{dr}\right] \tag{4.8}$$

It was assumed above that the detonation wave travels along a characteristic, therefore

$$a_2(r) = a[r, 0, u_{r2}(r)] = D(r) \tag{4.9}$$

Considering (4.7), we determine from this

$$u_{r2} = -1/(\gamma+1) \tag{4.10}$$

From (4.8) and (4.10) we get

$$b_2(r) = b[r, 0, u_{r2}] = -\frac{\nu\gamma}{(\gamma+1)^2}\frac{D^2}{r} - \frac{2}{\gamma+1}D\frac{dD}{dr} \tag{4.11}$$

For the characteristic which bounds the the quiescent gas we have

$$u_{t2}dt + u_{r2}dr = du = 0, \qquad \text{or} \qquad u_{t2} \pm a_2 u_{r2} = 0 \tag{4.12}$$

Substituting (4.12) into (4.4) and using (4.10) we have

$$\mp u_{r_2} da_2 = b_2 dt \qquad (4.13)$$

Considering (4.9) to (4.11) and (4.4), we obtain that along the sought-for characteristic

$$3\frac{dD}{D} + \frac{\nu\gamma}{\gamma+1}\frac{dr}{r} = 0 \qquad (4.14)$$

Integrating this equation and noting that the velocity, being directed toward the center, is taken as negative, we get

$$|D| = \frac{A}{r_2^m} \qquad \left(m = \frac{\nu\gamma}{3(\gamma+1)}\right), \quad \text{or} \quad -\frac{dr_2}{dt} = \frac{A}{r_2^m}$$

Integrating the last equation with the condition that $r_2 = 0$ at $t = 0$, we have

$$r_2 = a_n(-t)^n \qquad \left(n = \frac{1}{m+1} = \frac{3(\gamma+1)}{3(\gamma+1)+\nu\gamma}\right) \qquad (4.15)$$

Thus the law for the converging detonation front is obtained in the form which was looked for in Section 2, and furthermore the relation between the similarity exponent n, and γ and ν is determined.

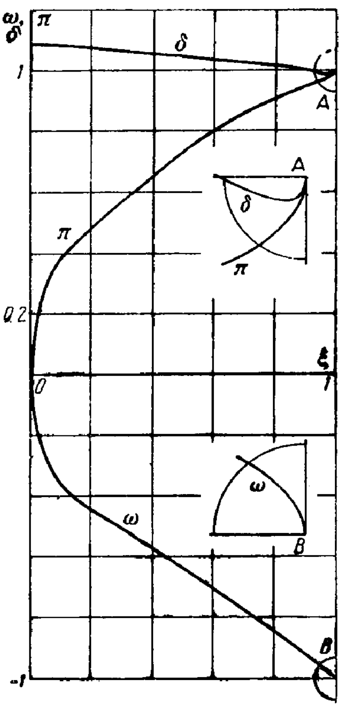

Fig. 2

5. Having determined $n(\gamma, \nu)$, we can solve the system (2,6) numerically with the boundary conditions (2,7). However, it is first necessary to examine the character of the point $\lambda = 1$, $\omega = -1$, $\pi = 1$, $\delta = 1$ which is the singular point, because the boundary conditions on the detonation wave are specified along the characteristic.

As an example, one of the possible solutions is shown in Fig. 2, which was obtained under the condition

$$d\lambda / d\omega = 0 \quad \text{for } \lambda = 1 \qquad (\xi = 1/\lambda) \quad (5.1)$$

Modification of condition (5.1) effects the integral curves very little, therefore the nature of the singular point was not considered in the present paper.

6. We compare the law obtained for the converging detonation front (for brevity it will be called "the solution with Q"), when Q is taken too large in the process of the acceleration ($\alpha = 1$), and the convergence law in the Guderley-Landau-Staniukovich solution, where Q is not considered at all, i.e. the solution is strictly valid only for $Q = 0$ ($\alpha = 2$). In reality the value of α changes during the process of convergence and $1 \leq \alpha \leq 2$. From this point of view the results of Section 5 (the distribution of v, p, ρ behind the front) can be compared with the solution of Guderley-Landau-Staniukovich.

The similarity exponent n for the solution with Q and for the Guderley-Landau-Staniukovich solution in the case of cylindrical symmetry ($\nu = 1$) for $\gamma = 1.4$ is 0.838 according to (4.15) and 0.834 according to [4], and for $\gamma = 3$ is respectively 0.800

and 0.810. Similarly, in the case of spherical symmetry ($\nu = 2$) for $\gamma = 1.4$, n is 0.720 and 0.717 and for $\gamma = 3$ it is respectively 0.667 and 0.638.

From this data it is evident that the difference between the similarity exponents (convergence rules) in the Guderley-Landau-Staniukovich solution and the obtained solution with Q (for the region in which there occurs a real mean convergence rule), is sufficiently small (especially for the case of cylindrical symmetry, $\nu = 1$). Thus, even if the released chemical energy increases toward the center according to the rule $1/r_2^{2m}$ this little affects the convergence rule. When Q is constant at the front its effect on the value of n in the convergence rule $r_2 = a_n(-t)^n$, is even less.

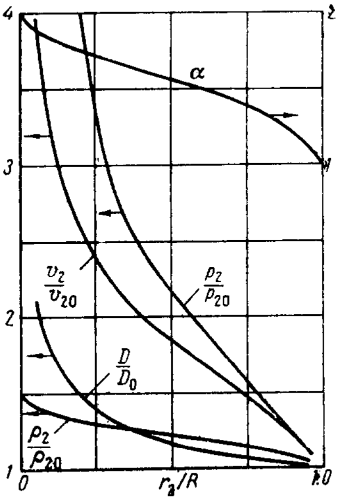

Fig. 3

The strongest effect on the convergence rule for the detonation front is due to the geometry (ν), a smaller effect is due to the fluid exponent γ, and the effect of the chemical energy released in the wave front is weak; Q determines the value of v, p, ρ in the detonation front (1.1) and (2.1).

From the foregoing it seems possible to use the following approach to the non-self-similar problem of the converging detonation front, started at certain initial radius R, and which has initiated its own motion at the Chapman-Jouguet velocity

$$D^2_0 = 2(\gamma - 1)(\gamma + 1) Q_0 \tag{6.1}$$

Taking for the exponent n or m a value from (4.15) for the corresponding γ and ν (self-similarity was not used in Section 4), we have the following relations for the motion of the front :

$$\frac{D}{D_0} = \left(\frac{R}{r_2}\right)^m, \qquad m = \frac{\nu\gamma}{\gamma + 1} \tag{6.2}$$

Substituting the last expression into (1.2) and using (1.1), we obtain successively

$$\alpha = 1 + \left[1 - \left(\frac{r_2}{R}\right)^{2m}\right]^{1/2}$$

$$\frac{v_2}{v_{20}} = \alpha \left(\frac{R}{r_2}\right)^m, \qquad \frac{p_2}{p_{20}} = \alpha \left(\frac{R}{r_2}\right)^{2m}, \qquad \frac{\rho_2}{\rho_{20}} = \frac{\gamma}{\gamma - (\alpha - 1)} \tag{6.3}$$

Here v_{20}, p_{20}, ρ_{20} are the values corresponding to the values in the Chapman-Jouguet detonation regime for the specified Q_0. The results for $\gamma = 3$, $\nu = 1$ are shown in Fig. 3.

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SOLUTION OF HELMHOLTZ' EQUATION FOR A HALF-PLANE WITH BOUNDARY CONDITIONS CONTAINING HIGH ORDER DERIVATIVES

(O RESHENII URAVNENIIA GEL'MGOL'TSA DLIA POLUPLOSKOSTI PRI GRANICHNYKH USLOVIIAKH, SODERZHASHCHIKH PROIZVODNYE VYSOKOGO PORIADKA)

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The problem of the steady-state acoustic oscillation is examined in a fluid the surface of which is covered by an infinitely thin elastic body (a membrane, a plate, a shell). Properties of the cover are given by means of a differential operator of arbitrary order with constant coefficients. A solution of the problem is formulated for arbitrary sources (point or distributed) which are located both in the fluid and on the cover.

Notation

P — pressure, f — extraneous body force in the fluid, F — extraneous surface force, ρ — density of fluid, ρ_0 — density of covering material, μ — surface density of coverage, E — Young's modulus, σ — Poisson's modulus, T — membrane tension, $2h$ — thickness of coverage, ω — circular frequency, k — wave number in the fluid.

The time factor $e^{-i\omega t}$ is omitted everywhere.

1. Formulation of the problem. Examples. Problem related to the influence of thin elastic objects (membranes, plates, shells) on acoustic processes in a fluid are at present of urgent interest. Mathematical boundary value problems which arise in the investigation of such effects as a rule have a specific feature: differential operators which are involved in the definition of boundary conditions have a higher order than the order of the equation itself.

Let the lower half-plane $y > 0$ be filled with a compressible fluid. Processes in this fluid will be described in terms of pressure P. For $y > 0$ we shall assume that the